Linear Algebra in Economics: The Leontief Input-Output Model

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Goal: Optimize production levels to meet total demand, taking into account demand created by productive sectors of the economy

Preliminary Terms and Definitions

Productive sector: Any sector of the economy that produces a good or service

Open Sector: Any sector of the economy that merely consumes goods or services without producing them

Final Demand (d): The values of the goods and services demanded by the open sectors of the economy (i.e. consumer demand

Intermediate Demand: The values of the goods and services demanded by the productive sectors of the economy

Total Demand: The sum of final and intermediate demand

Unit Consumption Vector (c_1, c_2, … , c_n) : A list of input values from *n* sectors needed by a particular sector, per unit of output

Consumption Matrix (C): Matrix representation of the unit consumption vectors of every sector in the economy

Leontief's Model (Production Equation):

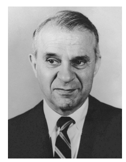
$$\mathbf{x} = \mathbf{C}\mathbf{x} + \mathbf{d},$$

Where **x** is a vector representation of the amount produced for each sector

This may also be written as:

$$(\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{d}$$

Where **I** is the identity matrix of $\mathbb{R}^n$, in an economy of n productive sectors



Wassily Leontief
https://upload.wikimedia.org/wikipedia/commons/3/36/Wassily_Leontief_1973.jpg

Necessary Linear Algebra Theory

- Create, understand, interpret vectors and matrices
  - Represent given data in these forms
- Vector and matrix algebra (addition, subtraction, multiplication)
- Construct the identity matrix for any $\mathbb{R}^n$
- Solve, manipulate vector equations
  - Namely, solving for **x** in Leontief's Model
- Row reduction to a Reduced Row Echelon Form (solving a linear system of equations)
- In some cases, inversion of a matrix is an alternative solution

Theorem 11 and its Consequences

Theorem 11: Alternate Form of Leontief's Model [1]

For a consumption matrix **C** of a given economy and final demand **d**, if C and d have non-negative entries and each column sum of C is less than 1, then $(\mathbf{I}-\mathbf{C})^{-1}$ exists, and

$$\mathbf{x} = (\mathbf{I}-\mathbf{C})^{-1} \cdot \mathbf{d}$$

Where x has non-negative entries and is the unique solution to Leontief's Model

[1] Lay, D. C., Lay, S. R., McDonald, J., (2016). 2.6 The Leontief Input-Output Model. In Linear algebra and its applications, fifth edition. Boston: Pearson.

Notes on Theorem 11:

- The prerequisite conditions will nearly always be met in any feasible, real-world application
- $(\mathbf{I}-\mathbf{C})^{-1} \approx \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \ldots + \mathbf{C}^m$, where $\mathbf{C}^m$ denotes the consumption matrix at the *m*th "round" of intermediate demand – $\mathbf{C}^m$ approaches the zero matrix relatively quickly, so this is a practical way to estimate $(\mathbf{I}-\mathbf{C})^{-1}$
- Entries of this matrix can be used to predict the change in production **x** when the final demand **d** changes

Real – World Application

The following table is based on actual data from the US economy in 1958:

**Units: millions of USD*	Units of Input Consumed per Unit of Output						
Purchased from:	**NMH**	**FMP**	**BMP**	**BNMPA**	**Energy**	**Services**	**MISC**
Nonmetal household and personal products (NMH)	0.1588	0.0064	0.0025	0.0304	0.0014	0.0083	0.1594
Final metal products (FMP)	0.0057	0.2645	0.0436	0.0099	0.0083	0.0201	0.3413
Basic metal products / mining (BMP)	0.0264	0.1506	0.3557	0.0139	0.0142	0.0070	0.0236
Basic nonmetal products and agriculture (BNMPA)	0.3299	0.0565	0.0495	0.3636	0.0204	0.0483	0.0649
Energy	0.0089	0.0081	0.0333	0.0295	0.3412	0.0237	0.0020
Services	0.1190	0.0901	0.0996	0.1260	0.1722	0.2368	0.3369
Entertainment/Misc. (MISC)	0.0063	0.0126	0.0196	0.0098	0.0064	0.0132	0.0012

Interpretation example: For 1 unit of Energy (i.e. to produce $1M in energy), the sector requires 0.0083 units (i.e. $8,300) worth of fine metal products

$$\mathbf{C} = \begin{bmatrix} .1588 & .0064 & .0025 & .0304 & .0014 & .0083 & .1594 \\ .0057 & .2645 & .0436 & .0099 & .0083 & .0201 & .3413 \\ .0264 & .1506 & .3557 & .0139 & .0142 & .0070 & .0236 \\ .3299 & .0565 & .0495 & .3636 & .0204 & .0483 & .0649 \\ .0089 & .0081 & .0333 & .0295 & .3412 & .0237 & .0020 \\ .1190 & .0901 & .0996 & .1260 & .1722 & .2368 & .3369 \\ .0063 & .0126 & .0196 & .0098 & .0064 & .0132 & .0012 \end{bmatrix}$$

Note: Each column is a *unit consumption vector*
The sum of the entries of each column is less than 1, and all entries are positive
Assuming all values in **d** are non-negative, this matrix is invertible and Theorem 11 holds

We know that $\mathbf{d} = (\mathbf{I}-\mathbf{C})\mathbf{x}$, and simplification of the matrix subtraction yields the following relationship between **x** and **d:**

$$\mathbf{d} = \begin{bmatrix} .8412 & -.0064 & -.0025 & -.0304 & -.0014 & -.0083 & -.1594 \\ -.0057 & .7355 & -.0436 & -.0099 & -.0083 & -.0201 & -.3413 \\ -.0264 & -.1506 & .6443 & -.0139 & -.0142 & -.0070 & -.0236 \\ -.3299 & -.0565 & -.0495 & .6364 & -.0204 & -.0483 & -.0649 \\ -.0089 & -.0081 & -.0333 & -.0295 & .6588 & -.0237 & -.0020 \\ -.1190 & -.0901 & -.0996 & -.1260 & -.1722 & .7632 & -.3369 \\ -.0063 & -.0126 & -.0196 & -.0098 & -.0064 & -.0132 & .9988 \end{bmatrix} \cdot \mathbf{x}$$

The Identity Matrix

$$I^n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$, where n is the number of columns

Solve for **x** using the following given **d**:

$$\begin{bmatrix} 74{,}000 \\ 56{,}000 \\ 10{,}500 \\ 25{,}000 \\ 17{,}500 \\ 196{,}000 \\ 5{,}000 \end{bmatrix} = \begin{bmatrix} .8412 & -.0064 & -.0025 & -.0304 & -.0014 & -.0083 & -.1594 \\ -.0057 & .7355 & -.0436 & -.0099 & -.0083 & -.0201 & -.3413 \\ -.0264 & -.1506 & .6443 & -.0139 & -.0142 & -.0070 & -.0236 \\ -.3299 & -.0565 & -.0495 & .6364 & -.0204 & -.0483 & -.0649 \\ -.0089 & -.0081 & -.0333 & -.0295 & .6588 & -.0237 & -.020 \\ -.1190 & -.0901 & -.0996 & -.1260 & -.1722 & .7632 & -.3369 \\ -.0063 & -.0126 & -.0196 & -.0098 & -.0064 & -.0132 & .9988 \end{bmatrix} \cdot \mathbf{x}$$

To solve for x, create an augmented matrix with d as the final column, and row reduce to the Reduced Row Echelon Form. This now becomes a simple linear system of equations.

$$\begin{bmatrix} .8412 & -.0064 & -.0025 & -.0304 & -.0014 & -.0083 & -.1594 & 74000 \\ -.0057 & .7355 & -.0436 & -.0099 & -.0083 & -.0201 & -.3413 & 56000 \\ -.0264 & -.1506 & .6443 & -.0139 & -.0142 & -.0070 & -.0236 & 10500 \\ -.3299 & -.0565 & -.0495 & .6364 & -.0204 & -.0483 & -.0649 & 25000 \\ -.0089 & -.0081 & -.0333 & -.0295 & .6588 & -.0237 & -.0020 & 17500 \\ -.1190 & -.0901 & -.0996 & -.1260 & -.1722 & .7632 & -.3369 & 196000 \\ -.0063 & -.0126 & -.0196 & -.0098 & -.0064 & -.0132 & .9988 & 5000 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 99575.65 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 97703.02 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 51230.52 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 131569.92 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 49488.49 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 329554.45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13835.34 \end{bmatrix}$$

Conclusion: in order to fulfill the given total demand, each sector must produce the following amounts (*in millions USD*):

NMH	FMP	BMP	BNMPA	Energy	Services	MISC
99576	97703	51231	131570	49488	329557	13835

Informational and data source: Lay, D. C., Lay, S. R., McDonald, J., (2016). 2.6 The Leontief Input-Output Model. In Linear algebra and its applications, fifth edition. Boston: Pearson.